Equilibria of Systems of Springs

Thomas Risse
Institute of Informatics & Automation, IIA
Faculty EEE&CS, Hochschule Bremen, Germany

LSBU, March 22nd 2012
1 The Problem

2 1D-Problem

3 2D-Problem
   Linear Approximation of the Elongation
   Quadratic Approximation of the Elongation
   Exact Modeling of the Elongation

4 Conclusion, Extensions and Generalisations
What is the Problem?

In answering to a request of one of my students let us look into the following problem:

There are some springs, connected in so called nodes, some of them are fixed in one end in some position. They become a system of springs, SOS. Initially all springs are released. Now, some external forces are applied to some of the nodes.

Determine the new equilibrium of the system, i.e. the new positions of the nodes.

In order to stepwise solve the problem

- reduce the dimension of the SOS
- use a simplified model of the restoring forces
- use examples, compare results with expectations, check plausibility, refine model
- perceive suitability of algorithms
Introductory 1D-Problem

Three springs are connected in two nodes $n_1$ and $n_2$; Two external forces $\vec{f}_1$ and $\vec{f}_2$ apply at $n_1$ and $n_2$ resp. All on one line: **one-dimensional** version of the problem

Let $x_i$ denote the displacement of node $n_i$, $x = (x_1, x_2)^T$. Then the elongations $e = (e_1, e_2, e_3)^T$ of the three springs are

$$ e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Ax $$

with **adjacency matrix** $A$. 
Introductory 1D-Problem

Hooke\textsuperscript{1} s law: strain is directly proportional to stress!
i.e. $\vec{f}_{\text{restoring}}$ is proportional to the elongation of the spring.

$$\begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \mathbf{e} = \mathbf{Ce} = \mathbf{CAx}$$

External forces are compensated by the restoring forces

$$\mathbf{f}_{\text{external}} = \mathbf{A}^\top \mathbf{f}_{\text{restoring}} = \mathbf{A}^\top \mathbf{CAx} = \mathbf{Kx}$$

with symmetric \textit{stiffness matrix} $\mathbf{K} = \mathbf{A}^\top \mathbf{CA}$. Solving the system of linear equations $\mathbf{Kx} = \mathbf{f}_{\text{external}}$ easily gives the unknown displacements $\mathbf{x}$.

\textsuperscript{1}R. Hooke (1635-1703)
Moving on to a 2D-Problem

As an example consider a SOS with 24 springs and 9 nodes.

Assume: all springs have unit length and unit spring constant; exactly one force applies to the center node.
Linear Approximation of the Elongation

In the example there are 12 horizontal and 12 vertical springs. A horizontal spring between \((x_1, y_1)\) and \((1 + x_2, y_2)\), and a vertical spring between \((x_1, y_1)\) and \((x_2, 1 + y_2)\) resp. have elongations

\[
e_h = e_h(x_1, y_1, x_2, y_2) = \sqrt{(1 + x_2 - x_1)^2 + (y_2 - y_1)^2} - 1
\]

and

\[
e_v = e_v(x_1, y_1, x_2, y_2) = \sqrt{(x_2 - x_1)^2 + (1 + y_2 - y_1)^2} - 1
\]

resp.

Taylor expansion gives the linear approximation\(^2\) of the elongation of a horizontal spring

\[
e_h = e_h(x_1, y_1, x_2, y_2) \approx x_2 - x_1
\]

and of a vertical spring analogously

\[
e_v = e_v(x_1, y_1, x_2, y_2) \approx y_2 - y_1
\]

\(^2\)c.p. e.g. H.Williamson, A.Mehta, M.Broussard, J.Cavazos, J.Bridge, S.Cox: The Physics of Springs

http://cnx.org/content/m32657/latest/
Linear Approximation of the Elongation

Again, we solve a system of linear equations $Kx = f_{\text{external}}$ where $x$ is the vector of displacements $(x_i, y_i)$ of node $i$.

We get an obviously not very convincing result: exact solution of an insufficient model does not yield a reasonable result!
Quadratic Approximation of the Elongation

Taylor expansion gives

\[ e_h = e_h(x_1, y_1, x_2, y_2) \approx x_2 - x_1 + \frac{1}{2}(y_2 - y_1)^2 \]

and analogously

\[ e_v = e_v(x_1, y_1, x_2, y_2) \approx y_2 - y_1 + \frac{1}{2}(x_2 - x_1)^2 \]

For \( n \) nodes with displacements \((x_i, y_i)\) let \( \mathbf{x} = (x_i)_{i=1,\ldots,n} \) and \( \mathbf{y} = (y_i)_{i=1,\ldots,n} \). \( \mathbf{A} \) is again the adjacency matrix. Then

\[ \mathbf{e} = \mathbf{A}*[\mathbf{x}; \mathbf{y}] + 1/2*(\mathbf{A}*[\mathbf{y}; \mathbf{x}]).^2 \]

\[ \mathbf{f}_{\text{restoring}} = \mathbf{C}^*\mathbf{e} \]

\[ \mathbf{f}_{\text{external}} = \mathbf{A}'*\mathbf{f}_{\text{restoring}} \]

\[ \mathbf{f}_{\text{external}} = \mathbf{A}'*\mathbf{C}^*(\mathbf{A}*[\mathbf{x}; \mathbf{y}] + 1/2*(\mathbf{A}*[\mathbf{y}; \mathbf{x}]).^2) \]

To solve this \textbf{non-linear} vector equation, the generic Nelder-Mead [5] algorithm, e.g. \texttt{fminsearch} applied to the scalar

\[ \text{norm}(\mathbf{f}_{\text{external}} - \mathbf{A}'*\mathbf{C}^*(\mathbf{A}*[\mathbf{x}; \mathbf{y}] + 1/2*(\mathbf{A}*[\mathbf{y}; \mathbf{x}]).^2)) \]

in 18 variables is not very reasonable!
Quadratic Approximation of the Elongation

Better to use the adequate Levenberg-Marquardt algorithm [2],[4],[6], e.g. `lsqnonlin` of the *optimization toolbox*.

The result is still not convincing: we do not take the direction of the restoring forces into account.
Exact Modeling of the Elongation

$f_{\text{restoring}}$ of a 'horizontal' spring with unit length and spring constant $c$ between $p = (x_1, y_1)$ and $q = (1 + x_2, y_2)$ is

$$f_{\text{restoring}} = c \text{ elongation} \frac{q-p}{|q-p|} = c(|q - p| - 1) \frac{q-p}{|q-p|}.$$
Conclusion, Extensions and Generalisations

*lessons learnt*

- Modeling is of importance!
- Only adequate algorithms produce reasonable results!
- By far not all web resources are reliable!

*suggested extensions*

- different positions, lengths, spring constants
- different initial conditions: prestress
- 3D

*suggested generalisations*

- dynamics: system behaviour in time – system of ode’s
- discrete $\rightarrow$ continuous – pde
References

www.ec-securehost.com/SIAM/FR18.html


Squares Problems; IMM, Technical University of Denmark 2004
www2.imm.dtu.dk/pubdb/views/edoc_download.php/3215/pdf


Minimization; Computer Journal 1965, Vol 7, 308-p313

www.ananth.in/docs/lmtut.pdf

Hochschule Bremen
www.weblearn.hs-bremen.de/risse/papers/LSBU